

# Is Zero a Natural Number?

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## 1 Introduction

The simplest way to answer the question in the title is to say that zero by definition is a natural number, or, as other do, to say that zero by definition is not a natural number. In mathematics we are allowed to make the definitions as we like so it is just a matter of definition and there is nothing more to say about it.

If I agreed on that this would be a very short article, but if we think of mathematics as a set of mental and conceptual tools to solve real world problems, some definitions are more useful than others and then it makes sense to discuss the consequences of different definitions, and which definitions are the most useful ones. The most basic notions of numbers are the cardinal and the ordinal numbers. Whether zero should be considered as a natural number depends on whether we would like to identify the natural numbers as the finite cardinal numbers or with the finite ordinal numbers. Let us take a closer look at these two classes of numbers.

## 2 Cardinal numbers

Many languages have several classes of numbers with cardinal numbers as one of them. Although counting has been an important activity for thousands of years and is one of the most basic models in mathematics, it was not before Cantor that a well-founded mathematical understanding emerged. The basic definition is that two sets are equipotent if there is a one-to-one pairing of the elements. It is easy to see that we get an equivalence relation. It is often inconvenient to use expressions like "I have found some mushrooms and the set of mushrooms is equipotent to the set of poles in our tent" because maybe the one that hears this message does not know the tent that is referred to that well. Although it is often inconvenient, the method is still in use. For instance: make one prayer for each pearl on the rosary. This makes sense without any counting as long as you have a rosary. The next step is to use some standard set to refer to. This is often related to the fingers and perhaps also the toes, because this set is normally available and these standard sets are equipotent for most humans.

Most people have some more or less developed standard sets that they use to understand the higher numbers. For instance 300 is for me is approximately the number of pupils in a Danish gymnasium. In books on science one also finds expressions like "This is of the same order as the number of atoms in the universe".

The next step is to have standard labels for sets of each equipotency class. Thus "five" is the label we use for all sets with five elements. From this point of view it does not make much sense to ask what "five" is. It is just a word that we use when we want to tell that a set has as many elements as we have fingers on one hand.

Most modern languages are far from fully developed in the use of such standard labels. For instance in English one can either say "one man" or "a man" where both indicate the same number, but "a" is normally not considered as a numeral. Some languages distinguish between different classes of objects and use different numerals for different classes. In some languages the nouns have different genders and the numeral change depending on the gender. In many languages the numerals interact with the inflections system of the language. Many languages also have inflections in singular or plural. In some languages there is even an inflection called dualis that is used when referring to two objects.

The empty set is obviously a set so it also needs a label and in English it is "zero". Like the other numerals the full understanding of zero is far from integrated in most languages. For instance in English singular is used when there is one object and plural when there are more. Zero is not singular so it must be plural, i.e. one man, two men, and zero men. There are other ways of indicating zero and one can both say there was no man and there was no men which in principle should be the same but in practice there is a light difference in meaning. These are indications that the concepts of numerals and of zero has not been fully incorporated in the language. For transfinite cardinal numbers it is worse, but this is not the topic of this article.

The emergence of the concept of zero has a long development. It may have started as an indication of an empty column on an abacus in ancient China. When writing a number a blank seem to have been used to indicate an empty column [4, 3]. Only later a symbol for an empty space was used and the first evidence of using a symbol for zero comes from India. Since then it has become an integrated part of mathematics. The use of symbols better reflects modern standards of mathematics than the languages that evolve quite slowly.

### 3 Ordinal numbers

When children start to learn numbers they do not distinguish between cardinal numbers and ordinal number. The distinction only develops gradually and some never really learn the difference. Still the difference between cardinal numbers and ordinal numbers is far from well implemented in the language. For instance in British English there is ambiguity of writing 18 January or 18<sup>th</sup> January although in both cases it should be clear that one refers to the 18<sup>th</sup> day of

a month called January rather than 18 month all called January. Sometimes one also hear adults misusing ordinal number by saying that one competitor perform double as bad as some other competitor in a competition because one of the competitor became no. 6 and the other became no. 3.

Let us take a look on the mathematical basis of ordinals. For two well-ordered sets  $A$  and  $B$  there either exists an injective mapping from  $A$  to  $B$ , or from  $B$  to  $A$ , such that the mapping maps left sections into left sections. This mapping is unique so if we have a large well-ordered set any element in any smaller well-ordered set is mapped into a specific element in the large set. In order to communicate an ordering we just have to communicate the mapping into the larger set. It is convenient to have a fixed well-ordered set and consider ordinal numbers as a well-ordered set of labels.

Therefore what people normally understand as ordinal numbers is a well ordered class of labels that are used to label elements in a well ordered set. We learn this as children. First we learn by heart a sequence one, two, three,  $\dots$  and then we learn to count objects in a set using this sequence. When counting we make a well-ordering of the objects and assign labels from the sequence to the objects. When we reach the last element in the set we automatically have the label for the cardinality of the set. As counting is such an important activity it is convenient that we reuse the cardinal numbers as ordinal numbers but since cardinal numbers and ordinal numbers are used for different purposes we modify the cardinal so that four become forth and eleven become eleventh. Clearly the names of these labels are derived from the names of the corresponding cardinal numbers except for “first” and “second”. This indicates that the conceptual distinction between cardinal and ordinal number is a much more recent conceptual invention than the cardinal numbers themselves. Obviously there is no need for the label “zeroth” in this system because an empty set has no element that should be assigned a label like “zero” or “zeroth”.

By experience we learn that counting leads to the same number independently of the what well ordering was chosen. Only mathematicians would ever realize that this could require a proof. For infinite sets the idea of counting brakes down so what to do in this case. The first idea might be simply to use the same labels for cardinal numbers and ordinal number, but for infinite sets this is not possible. One problem is that we do not know if the cardinal numbers are totally ordered. Only if we assume the axiom of choice we can be sure that any two cardinals are compatible. Another problem is that we do not know what the next cardinal number is after  $\aleph_0$  because this question is equivalent to the continuum hypothesis. Therefore an attempt to use the cardinals as labels would give serious restrictions to the foundation of mathematics because adding the axiom of choice and the continuum hypothesis or some versions of their negations have non-trivial consequences. These problems show that cardinal and ordinal numbers are different types of objects and that there is no reason to try to “reuse” cardinal numbers as ordinal numbers except if this is convenient.

It should be noted that people often use other labels to indicate orderings. In sports, for instance, one use the well-ordered set {gold, silver, bronze} for

the three best competitors. Integers are used to label other ordered sets as the set of Fourier coefficients  $\dots c_{-2}, c_{-1}, c_0, c_1, c_2, \dots$  for the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

The theorem that for any two well-ordered sets there exists a unique order isomorphism from a section of one of the sets to a left section of the other set, cannot be extended to lattices or any other class orderings. For instance there are many different order isomorphisms between the ordered set

$$(\dots c_{-2}, c_{-1}, c_0, c_1, c_2, \dots)$$

and  $\mathbb{Z}$ . For special purposes there may be special reasons to use a special enumeration. To start with 0 is not more natural than to start with 8 or  $-5$ . Sometimes there are even reasons to use non-integers. For instance quantum particles associated with irreducible representations of the rotation group are labeled by their spin type that is one of the numbers  $0, 1/2, 1, 3/2, \dots$ . Such special labels for a special purpose tells nothing about what should be the default labeling. There is obviously nothing wrong about using other sets than the ordinal numbers for labels but if we use the ordinal numbers as labels we need to know exactly what the set of ordinal numbers is. Words like “zeroth” or “minus first” that are sometimes used, but they indicate that the speaker has a deviating opinion about what the ordinal numbers are. This just dilutes the whole idea of having a fixed well-ordered set of elements that can be used as labels for elements in other well-ordered sets.

## 4 Order types

There exists a preordering on well-ordered set where we define A to be less than B if there exists a ordermorphism of A into B. The equivalence classes are called order types. Cantor wanted labels for the order types in parallel with the cardinal numbers that were as labels assigned to equipotency classes. The ordering of order types is a well ordering so one may reuse the labels for elements in a well-ordered set as labels for order types.

So here is the confusion. Ordinary people mainly use ordinal numbers to label *elements* of well-ordered sets but in enthusiasm of his newly developed sets theory Cantor [1] introduced ordinal numbers primarily as labels for *order types*. For each element in a well-ordered set A one can assign a well ordered set but there are two ways to do so. One is to use the set  $\{x \in A \mid x < a\}$  and another is to use  $\{x \in A \mid x \leq a\}$ . If the second convention is used there is no element corresponding to the empty set. If the first convention is used there is no element corresponding to the whole set. Clearly the problem is that a finite well ordered set with  $n$  elements has  $n + 1$  left sections and there is no way to completely solve this problem. Somehow the problem becomes kind of invisible if we work with well ordered sets because a set can always be embedded in a

larger whereas one cannot find a subset of the empty set. It was decided to associate the element  $a$  with the set  $\{x \in A \mid x < a\}$ . Hence the first element was associated with the empty set and the empty set has cardinality zero so among mathematicians zero has since then been considered as the first ordinal number in complete contrast with how people use ordinals to label elements of well ordered sets. If we use the second convention instead the first section is the empty set that has cardinality zero and the second section has cardinality one etc. The first convention may appear more natural for infinite sets but the correspondence between cardinals and ordinals breaks down anyway when we reach infinite sets where sections with ordinal types  $\omega$  and  $\omega + 1$  both have cardinality  $\aleph_0$ .

The order types have a simple additive structure. One order type is added to another order type by concatenating the corresponding well-ordered sets. For infinite sets this gives a non-commutative addition. Although addition makes sense for order types it makes no sense for elements of well-ordered sets. If you are the third in one competition and the fifth in a second competition you are not the seventh in any reasonable sense although this is what Cantor ordinal numbers suggest. The way to arrive at "seventh" is by associating third with the well-ordered set of the first and second in the first competition and then concatenate this well-ordered set with well-ordered set consisting of the first, the second, the third and the fourth of the second competition. This gives a well-ordered set of six elements and that is associated with a seventh element.

The real problem is that one wants to give labels to ordinal types rather than to elements of well ordered sets as was the original idea behind ordinal numbers. Actually I have never heard anybody outside mathematical institutes use an ordinal number as a label for a set. To do so would for instance be to use the word fifth about a ranking list of four persons of a competition. Normally we use the words "before" and "after" rather than "less" and "greater" when we refer to orderings like "the winner was Johnson and after him came MacDonald". I have never heard anybody saying that a ranking list with three elements is "before" a ranking list with five elements.

To use the the ordinal numbers as labels for order types of sets is clearly possible and perhaps relevant for special theoretical purposes but it has very little to do with what people normally call ordinal numbers.

## 5 Construction and further confusion

In the process of axiomatizing the foundation of mathematics the need of constructing numbers emerged. The idea is that we should avoid working with concepts that lead to inconsistencies. For instance after defining a Hilbert space it is good practice to give at least one example of a Hilbert space to ensure that all the consequences derived from the axioms and definitions do not lead to contradictions. In the attempts to axiomatize mathematics it was therefore needed to specify a sequence of labels that could be used to label equipotent

sets. Using basic set theoretic constructions the following sequence was derived:

$$\begin{aligned} &\emptyset \\ &\{\emptyset\} \\ &\{\emptyset, \{\emptyset\}\} \\ &\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\ &\vdots \end{aligned}$$

The idea was then to *define* 3 as the label  $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ . As proved by Gödel the use of such constructions is a not sufficient to ensure consistency of mathematics. It is remarkable that the idea has survived although it has completely failed and can only be used to prove that there exists a set with say three elements if anybody should doubt that. Although many textbooks still introduce the numbers in this way most mathematicians would still count it as an error if a student wrote  $2^3 \subseteq 3^2 + 2$  in an answer to an exercise. The point is that cardinal numbers are primarily labels rather than sets and a special language should be used for these labels (in this case the use of  $\subseteq$  rather than  $\subseteq$ ).

John Conway introduced a different construction of numbers based on combinatorial game theory [2]. If playing games were a more important activity than counting and ordering then his construction should definitely be the basis of our notion of numbers. Conway constructs something that he calls ordinal numbers in essentially the same way as above. Nevertheless the additive structure on Conway ordinals is different from the addition defined by concatenation of well-ordered sets. For instance Conway's addition is commutative whereas addition of ordinal number by concatenation is not. The most obvious conclusion is that Conway ordinals are related to but not the same as what is normally called ordinal numbers.

## 6 Consequences

The conclusion is that it is most natural to consider “zero” as the first cardinal number and “first” as the first ordinal number. Mathematicians should simply not confuse order types with ordinal numbers. Therefore mathematicians and computer scientists should not by default enumerate like  $x_0, x_1, x_2, \dots$ . Similar one should talk about a “polynomial of order zero” rather than a “zeroth order polynomial”.

This little story may also tell us why zero was invented much later than one, two, and three. The understanding of zero requires an understanding of the important distinction between cardinal numbers and ordinal numbers. When counting we make an ordering of the set and assign labels that are in principle ordinal numbers. When all elements have been labeled we translate the last ordinal number into a cardinal number. Hence zero was more difficult to understand than other cardinal numbers because it did not correspond to any ordinal number.

## References

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